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Geometry of moduli spaces of stable sheaves on rational Fano varieties

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2. D. A. Vassiliev. An Infinite Series of Rational Components of the Moduli Space of Rank 3 Sheaves on $\mathbb{P}^3$. Siberian Mathematical Journal. 2023. Vol. 64. No. 3. P. 525-541.

3. D. A. Vasil'ev, A. S. Tikhomirov. Moduli of rank two semistable sheaves on rational Fano threefolds of the main series. Mat. Sb., **215**:10 (2024), 3–57.

General description of the field of study

The vector bundles of rank two on the projective space $\mathbb{P}^3$ have been one of central objects of interest in algebraic geometry since 1970's, when it was discovered that certain algebraic rank 2 bundles on $\mathbb{P}^3$ are related to physical «instantons», which are defined to be anti-self-dual connections on the sphere S^4 with structure group SU(2) [5]. These bundles were called mathematical instantons. The study of moduli spaces of rank 2 bundles on $\mathbb{P}^3$ received more attention from 2010's, when it was shown that moduli spaces of mathematical instantons with fixed Chern classes are irreducible [34, 35]. However, the moduli spaces of general semistable coherent sheaves with fixed Chern classes may have several irreducible components, and their geometry is far from being well-understood. The study of sheaves of rank higher than 2 on $\mathbb{P}^3$ and the study of rank 2 sheaves on other Fano threefolds is only starting to be developed in last years (e.g., see recent articles [3, 12, 33]).

Main results of the thesis

In our thesis we construct several infinite series of irreducible moduli components of rank 2 coherent sheaves on rational Fano threefolds — the projective space $X_1 = \mathbb{P}^3$, the smooth quadric X_2, intersection of two quadrics X_4 and the codimension 3 linear section X_5 of the Grassmannian $\mathrm{Gr}(2,5)$ embedded by Plücker. They include series Σ_0 and Σ_1 from [36], series $M_{k,m,n}$ from Theorems 4.2 and 4.3 of [40], series $\widetilde{M}_m$ from Theorem 4.4 *loc. cit.*, series M_m from Theorem 4.5 *loc. cit.* and series $M_{k,m,n}$ from Theorem 4.6 *loc. cit.* Also we construct the series of irreducible components $\mathcal{S}_3(b,c)$ of moduli spaces of rank 3 coherent sheaves on $\mathbb{P}^3$ (see Assertion 2 of [39]). We prove that the components $\mathcal{S}_3(b,c)$ are rational if $3 \mid (2b+c)$ (see Theorem 2 of [39]). Also we prove that the components $\mathcal{S}(0,b,c)$ from [21] are rational (Theorem 3 of [39]), as well as the components from Theorems 4.2, 4.3, 4.4 and 4.5 of [40] and the components from Theorem 4.6 *loc. cit.* for varieties X_1, X_2 and X_5.

For the quadric X_2 we give exact bounds on the third Chern class c_3 of a semistable rank 2 sheaf with fixed c_1 and c_2 and classify semistable sheaves with maximal c_3 (see Theorem 3.1 of [40]). In this we follow the work of Schmidt [31] who studied analogously the sheaves on the projective space. A notable new result is the discovery of the first known example of a disconnected moduli space of rank 2 semistable sheaves with fixed Chern classes on a smooth projective threefold, see Theorem 5.4 of [40]. Also we give bounds on the third Chern class of so-called sheaves of main type on X_4 and X_5. (see Thorems 6.1 and 6.2 *loc. cit.*)

Place of the results in the general context of the field of study

Our description of irreducible components of moduli spaces of rank 2 bundles on $\mathbb{P}^3$ is a continuation of earlier results of other matematicians, in particular, of the results from [34, 35, 29, 14, 2]. We describe this in more detail in Section 2. The construction of irreducible components $\mathcal{S}_3(b,c)$ is a development of the idea of construction of components $\mathcal{S}(a,b,c)$ from [21], and we describe this in Section 4. Our description of semistable sheaves on the quadric X_2 with maximal third Chern class is a direct generalization of results, obtained by Schmidt for $\mathbb{P}^3$ in [31], see also Sections 5 and 6.

Methods of obtaining the results of the thesis

Irreducible components of moduli spaces of rank 2 bundles on $\mathbb{P}^3$ are obtained by the method of monads, introduced in [7]. The monads used are a generalization of monads from [2] and also use the notion of symplectic instanton bundles. Also the construction uses the methods from deformation theory. The proof of rationality of irreducible components $\mathcal{S}_3(b,c)$ is based on the results of Bialynicki-Birula [6] and uses the notion of equivariant resolution of singularities [23]. The description of semistable sheaves on X_2 with maximal third Chern class is obtained by means of the theory of Bridgeland stability conditions on threefolds and tilt stability, developed in [8, 32].

Possible applications of our results

Our study has a theoretical character. The results of it can be used for a further study of stable and semistable sheaves on Fano varieties, and, in particular, for obtaining exact bounds on the third Chern class of semistable rank two sheaves on X_4 and X_5.

Structure of the summary

The summary of the thesis is organised as follows. In Section 1 we recall the definition of moduli spaces of semistable sheaves and give some other preliminary notions and conventions. In Section 2 we describe our joint results with A. S. Tikhomirov and S. A. Tikhomirov [36] on the construction of two new infinite series of moduli components of rank 2 bundles on $\mathbb{P}^3$. Section 3 is based on results of Bialynicki-Birula which will be used in the next section. In Section 4 we describe our construction of an infinite series of rational moduli components of stable rank 3 bundles on $\mathbb{P}^3$ from [39]. In Section 5 we recall some concepts related to stability conditions of objects in derived categories and give some background material for the next section. In Section 6 we describe our joint results with A. S. Tikhomirov [40] on description of moduli spaces of rank 2 semistable sheaves on Fano varieties X_i and give bounds on the third Chern class of such sheaves.

1 Moduli spaces of semistable sheaves

The moduli spaces of coherent sheaves were first constructed by Mumford [27] in the case of vector bundles over curves using the notion of slope stability. This notion can also be given for sheaves over higher-dimensional varieties. Let X be a smooth n-dimensional projective variety over an algebraically closed field $\Bbbk$ of characteristic 0 with a very ample line bundle $\mathcal{O}_X(1)$ corresponding to a divisor H.

Definition 1. *The slope of a coherent sheaf E on X is defined as $\mu(E) = \frac{H^{n-1}\cdot c_1(E)}{H^n\cdot \mathrm{rk}(E)}$, where $c_1(E)$ denotes the first Chern class of E and $\mathrm{rk}(E)$ is the rank of E. Here dividing by zero is interpreted as $+\infty$.*

Definition 2. *A sheaf E is said to be μ-stable (respectively, μ-semistable) if for all proper subsheaves $0 \neq F \subset E$ the inequality $\mu(F) < \mu(E/F)$ (respectively, $\mu(F) \leq \mu(E/F)$) holds.*

The moduli spaces of sheaves on higher-dimensional varieties were constructed by Gieseker [16] and Maruyama [25, 26] using another notion of stability. With the same notation as above denote by $E(m)$ the coherent sheaf $E \otimes_{\mathcal{O}_X} \mathcal{O}_X(m)$.

Definition 3. *The Hilbert polynomial of E is defined as $P(E, m) = \chi(E(m))$, where $\chi(E(m))$ is the Euler characteristic of the sheaf $E(m)$.*

Let $f, g \in \mathbb{R}[m]$ be nonzero polynomials. If $\deg(f) > \deg(g)$, we set $f < g$. If $\deg(f) = \deg(g)$ and a, b are the leading coefficients in f, g, respectively, then we set $f < (\leq) g$ if $\frac{f(m)}{a} < (\leq) \frac{g(m)}{b}$ for all $m \gg 0$.

Definition 4. *A coherent sheaf E is called Gieseker (semi)stable, or simply (semi)stable, if for any proper subsheaf $0 \neq F \subset E$ we have $P(F, m) < (\leq) P(E/F, m)$.*

We will use later another notion of stability of sheaves, intermediate between μ-stability and Gieseker stability. Let now $\dim X = 3$. For a coherent sheaf E on X we define the numbers $a_i(E)$ for $i \in \{0, 1, 2, 3\}$ by $P(E, m) = a_3(E)m^3 + a_2(E)m^2 + a_1(E)m + a_0(E)$. Then we set $P_2(E, m) = a_3(E)m^2 + a_2(E)m + a_1(E)$.

Definition 5. *The sheaf E is called Gieseker 2-(semi)stable if for any proper subsheaf $0 \neq F \subset E$ we have $P_2(F, m) < (\leq) P_2(E/F, m)$.*

Stability, 2-stability and μ-stability of a sheaf are related by the following implications:

$$\begin{array}{ccccc} \mu\text{-stability} & \Longrightarrow & 2\text{-stability} & \Longrightarrow & \text{stability} \\ & & & & \Downarrow \\ \mu\text{-semistability} & \Longleftarrow & 2\text{-semistability} & \Longleftarrow & \text{semistability} \end{array}$$

For the definition of moduli spaces we will need the following notions.

Definition 6. *Let E be a semistable sheaf. A Jordan-Hölder filtration of E is a filtration*

$$0 = E_0 \subset E_1 \subset \ldots \subset E_n = E,$$

such that the factors $gr_i(E) = E_1/E_{i-1}$ are stable and for all i $P(gr_i(E), m) = c_i P(E, m), c_i \in \mathbb{R}$.

Proposition 1 ([20, Proposition 1.5.2])**.** *Jordan-Hölder filtrations always exist. The associated graded object $gr(E) = \oplus_i gr_i(E)$ does not depend on the choice of the Jordan-Hölder filtration.*

Definition 7. *Two semistable sheaves E_1 and E_2 with the same Hilbert polynomial are called S-equivalent if $gr(E_1) \cong gr(E_2)$.*

Let us recall the following category-theoretic notions. For a category $\mathcal{C}$ we denote by $\mathcal{C}^o$ the opposite category and by $\mathcal{C}'$ the functor category whose objects are functors $\mathcal{C}^o \to Sets$ and whose morphisms are natural transformations. There is a Yoneda functor which associates to an object $X \in \mathcal{C}$ a functor $\underline{X} : Y \to \mathrm{Mor}_{\mathcal{C}}(Y, X)$. The Yoneda functor embeds $\mathcal{C}$ as a full subcategory in $\mathcal{C}'$.

Definition 8. *A functor $\mathcal{F} \in \mathrm{Ob}\,\mathcal{C}'$ is corepresented by $F \in \mathrm{Ob}\,\mathcal{C}$ if there is a $\mathcal{C}'$-morphism $\alpha : \mathcal{F} \to \underline{F}$ such that any morphism $\alpha' : \mathcal{F} \to \underline{F}'$ factors through a unique morphism $\beta : \underline{F} \to \underline{F}'$. And $\mathcal{F}$ is represented by F if $\alpha : \mathcal{F} \to \underline{F}$ is an isomorphism.*

If an object F represents $\mathcal{F}$, then it also corepresents $\mathcal{F}$, and if F corepresents $\mathcal{F}$, then it is unique up to a unique isomorphism. We can rephrase the above definitions by saying that F represents $\mathcal{F}$ if $\mathrm{Mor}_{\mathcal{C}}(X, F) = \mathrm{Mor}_{\mathcal{C}'}(\underline{X}, \mathcal{F})$ for all $X \in \mathrm{Ob}\,\mathcal{C}$ and F corepresents $\mathcal{F}$ if $\mathrm{Mor}_{\mathcal{C}}(F, X) = \mathrm{Mor}_{\mathcal{C}'}(\mathcal{F}, \underline{X})$ for all $X \in \mathrm{Ob}\,\mathcal{C}$.

Let us now turn to the definition of our moduli functors. For a fixed polynomial $P \in \mathbb{Q}[z]$ define a functor $\mathfrak{M}' : Sch/\Bbbk \to Sets$ in the following way. For a $\Bbbk$-scheme S we define $\mathfrak{M}'(S)$ as the set of isomorphism classes of S-flat families of semistable sheaves on X with Hilbert polynomial P. The action of functor $\mathfrak{M}'$ a morphism $f : S' \to S$ is defined by pullback of families along the morphism $f \times \mathrm{id}_X$.

If $E \in \mathfrak{M}'(S)$ is an S-flat family of semistable sheaves, and L is line bundle on S, then $E \otimes p^*L$ (here $p : X \times S \to S$ is the canonical projection) is also an S-flat family, and these two families have isomorphic fibers over any point $s \in S$. It is therefore reasonable to consider the quotient functor $\mathfrak{M} = \mathfrak{M}'/\sim$, where $\sim$ is the following equivalence relation:

$$E \sim E' \text{ for } E, E' \in \mathfrak{M}'(S) \text{ iff } E \cong E' \otimes p^*L \text{ for some } L \in \mathrm{Pic}\, S.$$

A scheme $\mathcal{M}$ is called a moduli space of semistable sheaves if it corepresents the functor $\mathfrak{M}$.

Theorem 1 ([20, Theorem 4.3.4])**.** *There is a projective scheme $\mathcal{M}_{\mathcal{O}_X(1)}(P)$ that corepresents the functor $\mathfrak{M}$. Closed points in $\mathcal{M}_{\mathcal{O}_X(1)}(P)$ are in bijection with S-equivalence classes of semistable sheaves with Hilbert polynomial P.*

If the sheaves with Hilbert polynomial P on a threefold X with ample generator $\mathcal{O}_X(1)$ of its Picard group have rank r and Chern classes c_1, c_2, c_3, we will denote $\mathcal{M}_{\mathcal{O}_X(1)}(P)$ by $\mathcal{M}_X(r; c_1, c_2, c_3)$. If $r = 2$, then r will be sometimes dropped from the notation. Also we will denote by $\mathcal{B}_X(c_1, c_2)$ the open subset of $\mathcal{M}_X(2; c_1, c_2, 0)$ corresponding to stable locally free sheaves. And if $X = \mathbb{P}^3$, then the subscript X will be sometimes dropped.

In this thesis by a general point of irreducible scheme we mean a closed point belonging to some Zariski open dense subset of this scheme. Sometimes we will not make a distinction between a stable sheaf E and its isomorphism class $[E]$ as a point of moduli scheme.

For a coherent sheaf F on X and a non-negative integer n we will sometimes denote the sheaf $F^{\oplus n}$ by nF. We denote the sheaf cohomology groups $H^i(X, F)$ by $H^i(F)$.

2 Construction of rank 2 bundles on $\mathbb{P}^3$ via symplectic rank 4 bundles

In the description of the moduli spaces $\mathcal{B}(e, n)$ we can assume after twisting by line bundle that $e \in \{0, -1\}$. For $e = 0$ these moduli spaces are non-empty if $n \geq 1$, and for $e = -1$ if $n \geq 2$ is even [17].

It is now known [34, 35] that the scheme $\mathcal{B}(0, n)$ has an irredicible component I_n of expected (by deformation theory) dimension $8n-3$, and this component is the closure of a smooth open subset of I_n, consisting of the so-called mathematical instanton vector bundles. For the case $e = -1$ in [17, Exercise 4.3.2] Hartshorne constructed the first infinite series $\{\mathcal{B}_0(-1, 2m)\}_{m \geq 1}$ of irreducible components $\mathcal{B}_0(-1, 2m) \subset \mathcal{B}(-1, 2m)$, which have the expected dimension $16m - 5$.

Another infinite series of families of stable rank 2 bundles on $\mathbb{P}^3$, depending on triples of integers a, b, c, was described by Rao in 1984 [29] and in 1988 Ein independently described these families and proved that they constitute open subsets of irreducible components of $\mathcal{B}(e, n)$ [14].

Definition 9. *A monad [7] is a complex*

$$0 \to \mathcal{A} \xrightarrow{a} \mathcal{B} \xrightarrow{c} \mathcal{C} \to 0$$

of vector bundles, where a is an injective bundle map and c is surjective. In this situation the cohomology sheaf $E = \frac{\ker c}{\operatorname{im} a}$ is locally free.

Recall that a *symplectic structure* on a vector bundle E is an anti-self-dual isomorphism $\theta : E \xrightarrow{\sim} E^\vee, \theta^\vee = -\theta$, considered modulo proportionality.

Definition 10. *A symplectic vector bundle E on $\mathbb{P}^3$ is called a symplectic instanton [2], if*

$$h^0(E(-1)) = h^1(E(-2)) = h^2(E(-2)) = h^3(E(-3)) = 0,$$

$$c_2(E) = n \geq 1.$$

The number $c_2(E)$ is called the *charge* of an instanton E. The Beilinson spectral sequence implies that symplectic instantons of rank $2r$ and charge n are precisely the cohomology bundles of anti-self-dual monads of the form

$$0 \to n\mathcal{O}_{\mathbb{P}^3}(-1) \to (2n + 2r)\mathcal{O}_{\mathbb{P}^3} \to n\mathcal{O}_{\mathbb{P}^3}(1) \to 0. \tag{1}$$

In 2017 Ch. Almeida, M. Jardim, A. S. Tikhomirov and S. A. Tikhomirov [2] have constructed a new infinite series of irreducible components Y_a of spaces $\mathcal{B}(0, 1 + a^2)$ for $a \in \{2\} \cup \mathbb{Z}_{\geq 4}$. These components have dimensions $\dim Y_a = 4\binom{a+3}{3} - a - 1$, which for $a \geq 4$ are greater than expected. General sheaves from this components can be described as cohomology sheaves of monads, in which the middle term is a symplectic instanton of rank 4 and $c_2 = 1$ and the left and right terms are the sheaves $\mathcal{O}_{\mathbb{P}^3}(-a)$ and $\mathcal{O}_{\mathbb{P}^3}(a)$ respectively.

In a joint paper [36] with A. S. Tikhomirov and S. A. Tikhomirov we have constructed two new infinite series of irreducible components $\mathcal{M}(e, n)$ of spaces $\mathcal{B}(e, n)$, one for $e = 0$ and one for $e = -1$, which generalize the above construction from [2]. Namely, for $e = 0$ we constructed an infinite series Σ_0 of irreducible components $\mathcal{M}(0, n) \subset \mathcal{B}(0, n)$ such that a general bundle from $\mathcal{M}(0, n)$ can be described as a cohomology bundle of a monad of the form

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \to \mathrm{E} \to \mathcal{O}_{\mathbb{P}^3}(a) \to 0, \tag{2}$$

in which E is now a symplectic instanton of rank 4 with an arbitrary second Chern class and a is big enough.

In order to prove that the cohomology bundles of monads (2) form a dense open subset of an irreducible component of $\mathcal{B}(0, n)$, we consider a direct sum $\mathbb{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$ of two mathematical instanton bundles with charges $c_2(\mathcal{E}_1) = m \geq 1$ and $c_2(\mathcal{E}_2) = m + \varepsilon$, where $\varepsilon \in \{0, 1\}$. We show the vanishing of certain cohomology groups associated to general such bundles. The bundle $\mathbb{E}$ is a symplectic rank 4 instanton. This bundle and its deformations are used as middle terms of monads (2). We construct a universal family Y of cohomology bundles E of such monads and prove that the Kodaira–Spencer map from the tangent space of Y at a given point x_0 to $\mathrm{Ext}^1(E,E)$ is an isomorphism. Since Y is smooth at x_0, this implies that the image of Y in $\mathcal{B}(0,n)$ is indeed an open subset. We have $n = 2m+\varepsilon+a^2$.

The Hirzeruch–Riemann–Roch formula allows us to find the dimension of the space of symplectic instantons E, which is equal to $H^1(S^2\mathbb{E})$ by deformation theory, and the dimension of $H^0(\mathbb{E}(a)) = \chi(\mathbb{E}(a))$. We get that the dimension of $\mathcal{M}(0,n)$ is equal to $4\binom{a+3}{3} + (2m+\varepsilon)(10-a) - 11$.

In [36, Theorem 1] we have proved that the series Σ_0 contains components $\mathcal{M}(0,n)$ for all $n \gg 0$ (more precisely, at least for all $n \geq 146$). The series Σ_0 is the first known series after the instanton series $\{I_n\}_{n\geq 1}$ with such a property (for the Ein series the question whether it contains components for all big enough values of the second Chern class is open).

For the case $e = -1$ we analogously constructed a series Σ_1 of irreducible components $\mathcal{M}(-1,n)$ of spaces $\mathcal{B}(-1,n)$, where n is even, such that a general bundle from $\mathcal{M}(-1,n)$ is the cohomology bundle of a monad anologous to monads above, in which the middle term is a so-called twisted symplectic instanton bundle of rank 4 with the first Chern class equal to -2 and an arbitary even second Chern class, while the left and the right terms are now $\mathcal{O}_{\mathbb{P}^3}(-a-1)$ and $\mathcal{O}_{\mathbb{P}^3}(a)$, respectively, with a big enough. As a test twisted symplectic rank 4 bundle we consider a direct sum $\mathcal{E}_1 \oplus \mathcal{E}_2$, where $\mathcal{E}_1 \in \mathcal{B}_0(-1,2m), \mathcal{E}_2 \in \mathcal{B}_0(-1,2(m+\varepsilon)), \varepsilon \in \{0,1\}$. We have $n = 4m + 2\varepsilon + a(a+1)$. The dimension of this component is equal to $4\binom{a+3}{3} + 2\binom{a+3}{2} - (2m+\varepsilon)(2a-19) - 17$.

In [36, Theorem 2] we proved that Σ_1 contains components $\mathcal{M}(-1,n)$ for asymptotically all big enough even n. More precisely, if $\mathcal{N}$ is the set of all n for which there exist a component $\mathcal{M}(-1,n) \in \Sigma_1$, then

$$\lim_{r\to\infty} \frac{|\mathcal{N} \cap \{2,4,\ldots,2r\}|}{r} = 1.$$

Also in [36, §4] we have found all values $n \leq 20$ for which there exist a component $\mathcal{M}(0,n)$. We calculated their dimensions and spectra of general bundles from these components. And we made the same for all components $\mathcal{M}(-1,n)$ with $n \leq 40$.

3 Results of Bialynicki-Birula

In [39] we proved rationality of irreducible components of moduli spaces of stable sheaves on $\mathbb{P}^3$ from two infinite series (of rank 2 and rank 3 sheaves, respectively). The main ingredient of the proof is constituted by results of Bialynicki-Birula [6], which we recall in this section.

Take an algebraic group scheme G over $\Bbbk$ and an algebraic scheme Y over $\Bbbk$. Then $G \times Y$ has a natural group scheme structure over Y. Consider a finite-dimensional vector space V over $\Bbbk$ and a group homomorphism $\alpha : G \to GL(V)$.

Following [6], refer as a *trivial α-bundle over Y* to the Y-scheme $V \times Y$ with the action of $G \times Y$ induced by α. A Y-scheme X is an *α-bundle* if there exists an open covering $Y = \bigcup_i Y_i$ such that the fibered product $X \times_Y Y_i$ is isomorphic as a Y_i-scheme to the trivial α-bundle over Y_i for each i. A Y-scheme X with an action of $G \times Y$ is a *G-bundle* whenever there exists an open covering $Y = \bigcup_i Y_i$ such that $X \times_Y Y_i$ for each i is an α_i-bundle over Y_i with $\alpha_i : G \to GL(V_i)$. If $\dim V_i = n$ for each i then we say that the G-bundle is *of dimension n.*

From now on we put $G = \mathbb{G}_m$. If V is a vector space over $\Bbbk$ carrying a linear representation of G then denote by V^0 the subrepresentation consisting of all $v \in V$ with $G(\Bbbk) \cdot v = v$. Denote the subrepresentations that are the linear spans of $v \in V$ such that for $\lambda \in G(\Bbbk) \cong \Bbbk^\times$ the result of the action of λ on v equals $\lambda^m v$ for $m > 0$ and $m < 0$ by V^+ and V^-. We have $V = V^0 \oplus V^+ \oplus V^-$. Note that for the action of G on an algebraic $\Bbbk$-scheme X and for a closed $a \in X^G$ in the fixed-point set the tangent space $T_a(X)$ carries a canonical representation of G.

In the next proposition all algebraic schemes are reduced, while X is a nonsingular projective algebraic scheme with an action of G.

Proposition 2 ([6, Theorem 4.1]). *If $X^G = \bigcup_{i=1}^r (X^G)_i$ is the decomposition of X^G into connected components then for each $i = 1, \dots, r$ there exists a unique locally closed nonsingular G-invariant subscheme X_i^+, respectively X_i^-, of the scheme X, as well as a unique morphism $\gamma_i^+ : X_i^+ \to (X^G)_i$, respectively $\gamma_i^- : X_i^- \to (X^G)_i$, such that the following hold:*

(a) *$(X^G)_i$ is a closed subscheme of X_i^+, respectively X_i^-, and the morphism $\gamma_i^+|_{(X^G)_i}$, respectively $\gamma_i^-|_{(X^G)_i}$, is the identity;*

(b) *X_i^+, respectively X_i^-, with the induced action of G and the morphism γ_i^+, respectively γ_i^-, is a G-bundle over $(X^G)_i$;*

(c) *for every closed* $a \in (X^G)_i$ *we have*

$$T_a(X_i^+) = T_a(X)^0 \oplus T_a(X)^+, \qquad T_a(X_i^-) = T_a(X)^0 \oplus T_a(X)^-.$$

The dimension of the G*-bundle defined in* (b) *equals* $T_a(X)^+$*, respectively* $T_a(X)^-$*, for every closed* $a \in (X^G)_i$*.*

Moreover, $X = \bigcup_{i=1}^r X_i^+ = \bigcup_{i=1}^r X_i^-$ according to [6, Theorem 4.3].

Given a prescribed action $\eta : \mathbb{G}_m \times X \to X$ of $\mathbb{G}_m$ on a proper algebraic variety X and $p \in X$, the mapping $\eta(-,p) : \mathbb{A}^1 \backslash \{0\} \cong \mathbb{G}_m \to X$ extends uniquely to the regular mapping $\overline{\eta(-,p)} : \mathbb{A}^1 \to X$, and we will denote $\overline{\eta(-,p)}(0)$ by $\eta^0(p)$.

Our proof of rationality of moduli components is based on the following simple corollary of Proposition 2.

Lemma 1 ([39, Lemma 1])**.** *Consider a nonsingular projective variety* X *with an action* η *of the group* $G = \mathbb{G}_m$*. Suppose that for a dense open subset* $U \subset X$ *there exists a rational subvariety* $Y \subset X^G$ *such that* $\eta^0(u) \in Y$ *for every* $u \in U$*. Then* X *is rational.*

Indeed, in the situation of the lemma U is isomorphic to a dense open subset of a G-bundle over Y, hence U is birational to a product of Y with an affine space.

4 An infinite series of rational moduli components of rank 3 sheaves on $\mathbb{P}^3$

Recall that a sheaf F is called *reflexive* if the natural map $F \to F^{\vee\vee}$ is an isomorphism. Reflexive sheaves are in several aspects simpler to study than general coherent sheaves, for example, a reflexive rank 2 sheaf F on $\mathbb{P}^n$ with $c_1(F) \in \{-1, 0\}$ is stable if and only if $H^0(F) = 0$ [28, Chapter 2, Lemma 1.2.5].

In [21, Section 2.2] M. Jardim, D. Markushevich and A. S. Tikhomirov consider the morphisms

$$a\mathcal{O}_{\mathbb{P}^3}(-3) \oplus b\mathcal{O}_{\mathbb{P}^3}(-2) \oplus c\mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} (a+b+c+2)\mathcal{O}_{\mathbb{P}^3}$$

whose singular set

$$\Delta(\alpha) = \{x \in \mathbb{P}^3 \mid \alpha(x) \text{ is not injective}\}$$

is 0-dimensional. In this situation $\operatorname{coker}(\alpha)$ is a stable reflexive sheaf of rank 2. If $3a+2b+c=2k$ is even and positive; then, denoting the normalized sheaf $\operatorname{coker}(\alpha)(-k)$ by E, we obtain the exact sequence

$$0 \to a\mathcal{O}_{\mathbb{P}^3}(-3-k)\oplus b\mathcal{O}_{\mathbb{P}^3}(-2-k)\oplus c\mathcal{O}_{\mathbb{P}^3}(-1-k) \xrightarrow{\alpha} (a+b+c+2)\mathcal{O}_{\mathbb{P}^3}(-k) \to E \to 0, \tag{3}$$

where $c_1(E)=0$. Jardim, Markushevich and Tikhomirov showed in [21, Theorem 8] that the family of sheaves E appearing in the exact triples of the form (3) constitute a smooth dense open subset $\mathcal{S}(a,b,c)$ of an irreducible component of the moduli space of stable reflexive rank 2 sheaves on $\mathbb{P}^3$. For simplicity, we will call $\mathcal{S}(a,b,c)$ an irreducible component.

Our article [39] presents an analog of the above construction for rank 3 sheaves by considering the morphisms

$$b\mathcal{O}_{\mathbb{P}^3}(-2)\oplus c\mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} (b+c+3)\mathcal{O}_{\mathbb{P}^3}, \tag{4}$$

whose singular set $\Delta(\alpha)=\{x\in\mathbb{P}^3 \mid \alpha(x) \text{ is not injective}\}$ is either empty or 0-dimensional. Here $\operatorname{coker}(\alpha)$ is a reflexive sheaf of rank 3.

Consider the action $\eta_{\mathbb{P}^3}$ of the group $\mathbb{G}_m$ on $\mathbb{P}^3=\mathbb{P}(V)$, given in coordinates as

$$\eta_{\mathbb{P}^3}: \mathbb{G}_m\times\mathbb{P}^3 \to \mathbb{P}^3, \quad (t,(x_0:x_1:x_2:x_3)) \mapsto (x_0:tx_1:tx_2:tx_3).$$

Observe that the fixed points of this action are $a_0 := (1:0:0:0)$ and the points of the plane $H := \{x_0=0\}$.

The action $\eta_{\mathbb{P}^3}$ of $\mathbb{G}_m$ on $\mathbb{P}^3$ induces the action of $\mathbb{G}_m$ on the set of coherent sheaves on $\mathbb{P}^3$, given on the closed points as $E\mapsto t^*E$, where for $t\in\mathbb{G}_m(\Bbbk)$ we denote by the same letter the action of t on $\mathbb{P}^3$.

The stability of the sheaf $\operatorname{coker}\alpha$ from (4) is not obvious. There is a relatively simple criterion for μ-stability of reflexive rank 3 sheaves:

μ-stability Criterion ([28, Remark 1.2.6]). *A reflexive rank 3 sheaf E on $\mathbb{P}^n$ with $c_1(E)=0$, respectively $c_1(E)=-1,-2$, is μ-stable if and only if $H^0(\mathbb{P}^n,E)=H^0(\mathbb{P}^n,E^\vee)=0$, respectively $H^0(\mathbb{P}^n,E)=H^0(\mathbb{P}^n,E^\vee(-1))=0$.*

Suppose that $b,c\ge 0$, $k\ge 1$, $c_1\in\{0,-1,-2\}$, and $2b+c=3k+c_1$. Take a rank 3 sheaf E on $\mathbb{P}^3$ with the first Chern class c_1 fitting into the exact triple

$$0\to b\mathcal{O}_{\mathbb{P}^3}(-k-2)\oplus c\mathcal{O}_{\mathbb{P}^3}(-k-1) \xrightarrow{\alpha} (b+c+3)\mathcal{O}_{\mathbb{P}^3}(-k) \to E\to 0; \tag{5}$$

furthermore, the singular set $\Delta(\alpha)$ is either empty or 0-dimensional; as for rank 2 sheaves in [21, Section 2.2], this condition holds for general α.

We proved the following auxiliary result:

Theorem 2 ([39, Theorem 1]). *For all b and c but $(b,c) = (0,1)$, there exists a $\mathbb{G}_m$-invariant Gieseker stable rank 3 reflexive sheaf E fitting into an exact triple of the form (5) whose restriction to H is stable and locally free.*

For all cases except to $(b,c) = (0,3)$ and $(b,c) = (3,0)$ the proof goes by giving an explicit map $\alpha : b\mathcal{O}_{\mathbb{P}^3}(-k-2) \oplus c\mathcal{O}_{\mathbb{P}^3}(-k-1) \xrightarrow{\alpha} (b+c+3)\mathcal{O}_{\mathbb{P}^3}(-k)$ and proving that $E = \operatorname{coker}\alpha$ is μ-stable with the help of μ-stability criterion given above. In the remaining two cases we adapt the arguments from [38] to show that for a general α as above the sheaf $E = \operatorname{coker}\alpha$ is Gieseker stable.

One can show that the map $\operatorname{Hom}(b\mathcal{O}_{\mathbb{P}^3}(-2-k) \oplus c\mathcal{O}_{\mathbb{P}^3}(-1-k), (b+c+3)\mathcal{O}_{\mathbb{P}^2}(-k)) \to \operatorname{Ext}^1(E,E)$, induced by an exact triple (5), is surjective. Also we have $\operatorname{Ext}^2(E,E) = 0$. This implies the following statement:

Proposition 3 ([39, Assertion 2]). *The moduli space of Gieseker stable sheaves E in (5) is a smooth dense open subset $\mathcal{S}_3(b,c)$ of an irreducible component of the moduli space of Gieseker stable reflexive rank 3 sheaves on $\mathbb{P}^3$.*

The dimension of the component $\mathcal{S}_3(b,c)$ containing a point $[E]$ is equal to $12c_2(E)-8$ if $c_1(E) = 0$; respectively, $12c_2(E)-12$ if $c_1(E) = -1$; or $12c_2(E)-24$ if $c_1(E) = -2$.

As above, $b, c \geq 0$, $k \geq 1$, $c_1 \in \{0,-1,-2\}$, and $2b + c = 3k + c_1$. Suppose that $\mathcal{E}$ is a locally free sheaf on the projective plane $\mathbb{P}^2$ fitting into the an exact triple of the form

$$0 \to b\mathcal{O}_{\mathbb{P}^2}(-2-k) \oplus c\mathcal{O}_{\mathbb{P}^2}(-1-k) \xrightarrow{\alpha'} (b+c+3)\mathcal{O}_{\mathbb{P}^2}(-k) \to \mathcal{E} \to 0. \qquad (6)$$

Our proof of Theorem 2 also shows that a general such sheaf is Gieseker stable. An argument similar to that in the case of sheaves on $\mathbb{P}^3$ shows that the moduli space of stable sheaves $\mathcal{E}$ is a dense open subset of an irreducible component of the moduli space of stable rank 3 vector bundles on $\mathbb{P}^2$ with the first Chern class c_1; denote this subset by $\mathcal{Y}$.

Denote by $M_{\mathbb{P}^2}(k,n)$ the moduli variety of stable rank k vector bundles V on $\mathbb{P}^2$ with Chern classes $c_1(V) = 0$ and $c_2(V) = n$. We use the following result [22, Corollary 0.3.a]:

Proposition 4. *If* $(k, n) = 1, 2, 3, 4$ *then* $M_{\mathbb{P}^2}(k, n)$ *is rational.*

In our case $k = 3$ and $(k, n) \in \{1, 3\}$, so that the variety $M_{\mathbb{P}^2}(3, n)$ is rational for every n. Thus, $\mathcal{Y}$ is rational for $c_1(\mathcal{E}) = 0$, i.e., for $3 \mid (2b + c)$.

Theorem 3 ([39, Theorem 2])**.** *The variety* $\mathcal{S}_3(b, c)$ *is rational for* $3 \mid (2b + c)$.

Let us repeat here the main steps of the proof. For simplicity of notation put $\mathcal{S} := \mathcal{S}_3(b, c)$. Denote by $\mathcal{S}^0 \subset \mathcal{S}$ the subvariety of isomorphism classes of those sheaves from $\mathcal{S}$, which do not have singularities on $H = \{x_0 = 0\}$. By $\mathcal{S}^0_{\mathrm{inv}} \subset \mathcal{S}^0$ we denote the subvariety of isomorphism classes of $\mathbb{G}_m$-invariant sheaves from $\mathcal{S}^0$.

Lemma 2 ([39, Lemma 7])**.** *The variety* $\mathcal{S}^0_{\mathrm{inv}}$ *is rational for* $3 \mid (2b + c)$.

This lemma is proved by construction of mutually inverse morphisms between dense open subsets of varieties $\mathcal{S}^0_{\mathrm{inv}}$ and $\mathcal{Y}$. The exact definition of these morphisms uses the construction of moduli spaces of stable sheaves via Quot schemes.

The proof of rationality of $\mathcal{S}_3(b, c)$ for $3 \mid (2b+c)$ is finished by considering the projective closure $\mathcal{S} \subset \overline{\mathcal{S}}$, the equivariant resolution of singularities $\Pi : \overline{\mathcal{S}}_{\mathrm{sm}} \to \overline{\mathcal{S}}$ with an action $\eta_{\overline{\mathcal{S}}_{\mathrm{sm}}}$ of $\mathbb{G}_m$ and by invoking Lemma 1, since for the points x from an open dense subset of $\overline{\mathcal{S}}_{\mathrm{sm}}$ the corresponding points $\eta^0_{\overline{\mathcal{S}}_{\mathrm{sm}}}(x)$ belong to a variety, isomorphic to a dense open subset of the variety $\mathcal{S}^0_{\mathrm{inv}}$, which is rational.

Moreover, by the same method in [39, §5] we prove the rationality of the components $\mathcal{S}(0, b, c)$ of the moduli space of rank 2 sheaves on $\mathbb{P}^3$.

5 Stability of objects in derived categories

From now on we assume that the base field is the field of complex numbers.

In 1980, R. Hartshorne, investigating in [18] the spectra of stable reflexive coherent sheaves of rank two on $\mathbb{P}^3$, proved the boundedness of the third Chern class c_3 of these sheaves for fixed first and second Chern classes c_1 and c_2. The exact estimates he obtained for the class c_3 have the form (see [18, Thm. 8.2])

$$c_3 \leq c_2^2 - c_2 + 2, \quad \text{if} \quad c_1 = 0; \qquad\qquad c_3 \leq c_2^2 \quad \text{if} \quad c_1 = -1. \tag{7}$$

In the same work, the irreducibility, smoothness and rationality of the moduli spaces of such sheaves with $c_1 = -1$, arbitrary $c_2 > 0$ and maximal $c_3 = c_2^2$ are proved.

In 2018, B. Schmidt in [31], investigating the properties of tilt stability in the bounded derived category of coherent sheaves $\mathrm{D}^b(\mathbb{P}^3)$, proved that the estimates of (7) are true for all semistable sheaves of rank two on $\mathbb{P}^3$, and gave an explicit description of their moduli space for $-1 \le c_1 \le 0$, $c_2 > 0$ and maximal c_3. As a consequence, he obtained that these spaces are irreducible smooth rational projective varieties, except for one case, which was studied before in [38]. It is not difficult to see that the moduli spaces of reflexive sheaves described by Hartshorne are open subsets of these varieties. We also note that quite recently in the 2023 work [33], Schmidt generalized the above results to the case of sheaves on $\mathbb{P}^3$ of all ranks from 0 to 4.

In a joint work with A. S. Tikhomirov [40] we studied the moduli spaces of semistable rank two sheaves on rational three-dimensional Fano varieties of the main series. There are four such varieties — these are the projective space $X_1 = \mathbb{P}^3$, the three-dimensional quadric X_2, the complete intersection X_4 of two quadrics in the space $\mathbb{P}^5$, and the section X_5 of the Grassmannian $\mathrm{Gr}(2,5)$ embedded by Plücker in the space $\mathbb{P}^9$ by a linear subspace $\mathbb{P}^6$. Here the subscript i of the variety X_i is its projective degree.

Let us recall the concept of tilt stability, following the presentation in [31]. Let X be one of the varieties $X_i,\ i = 1,\ 2,\ 4,\ 5$. Cohomology ring $H^*(X,\mathbb{Z})$ is generated by the classes of a hyperplane section $H \in H^2(X,\mathbb{Z})$, a line $L \in H^4(X,\mathbb{Z})$ (understood as a projective line in the space $\mathbb{P}^{2+i} \supset X_i = X$ for $i = 1, 2$, respectively, $X_i \hookrightarrow \mathbb{P}^{1+i}$ for $i = 4, 5$) and a point $\{\mathrm{pt}\} \in H^6(X,\mathbb{Z})$ (for simplicity we will also denote the class of a point by 1).

Let $\beta \in \mathbb{R}$. Define *twisted Chern character* as $\mathrm{ch}^\beta = e^{-\beta H} \cdot \mathrm{ch}$. Let us present explicit formulas for the components $\mathrm{ch}^\beta_i = \mathrm{ch}^\beta_i(E)$:

$$\begin{aligned}
&\mathrm{ch}^\beta_0 = \mathrm{rk}(E),\ \mathrm{ch}^\beta_1 = \mathrm{ch}_1 - \beta H \mathrm{ch}_0,\ \mathrm{ch}^\beta_2 = \mathrm{ch}_2 - \beta H \mathrm{ch}_1 + \frac{\beta^2}{2} H^2 \mathrm{ch}_0,\\
&\mathrm{ch}^\beta_3 = \mathrm{ch}_3 - \beta H \mathrm{ch}_2 + \frac{\beta^2}{2} H^2 \mathrm{ch}_1 - \frac{\beta^3}{6} H^3 \mathrm{ch}_0.
\end{aligned} \tag{8}$$

Define a *torsion pair*

$$\mathcal{T}_\beta = \{E \in \mathrm{Coh}(X)\colon \text{any quotient } E \to G \text{ satisfies } \mu(G) > \beta\},$$

$$\mathcal{F}_\beta = \{E \in \mathrm{Coh}(X)\colon \text{any subsheaf } 0 \neq F \to E \text{ satisfies } \mu(F) \le \beta\}$$

and a category $\mathrm{Coh}^\beta(X)$ as the extension closure $\langle \mathcal{F}_\beta[1], \mathcal{T}_\beta \rangle$ in $D^b(X)$. For

$\alpha \in \mathbb{R}_+$ the *tilt-slope* of an object $E \in \mathrm{Coh}^\beta(X)$ is defined as

$$\nu_{\alpha,\beta}(E) = \nu_{\alpha,\beta}(\mathrm{ch}_0(E), \mathrm{ch}_1(E), \mathrm{ch}_2(E)) = \frac{H \cdot \mathrm{ch}_2^\beta(E) - \frac{\alpha^2}{2} H^3 \cdot \mathrm{ch}_0^\beta(E)}{H^2 \cdot \mathrm{ch}_1^\beta(E)}.$$

An object $E \in \mathrm{Coh}^\beta(X)$ is called to be *tilt-(semi)stable* (or $\nu_{\alpha,\beta}$-*(semi)stable*), if for any subobject $0 \neq F \hookrightarrow E$ we have $\nu_{\alpha,\beta}(F) < (\leq)\, \nu_{\alpha,\beta}(E/F)$.

The connection between tilt stability and Gieseker stability is provided by the following statement.

Proposition 5 ([40, Proposition 2.1 (i)]). *An object $E \in \mathrm{Coh}^\beta(X)$ is $\nu_{\alpha,\beta}$-(semi)stable for $\beta < \mu(E)$ and $\alpha \gg 0$ iff E is a 2-(semi)stable sheaf.*

Let us also recall the construction of Bridgeland stability conditions on X. Let

$$\mathcal{T}'_{\alpha,\beta} = \{E \in \mathrm{Coh}^\beta(X) \mid \text{any quotient } E \twoheadrightarrow G \text{ satisfies } \nu_{\alpha,\beta}(G) > 0\},$$
$$\mathcal{F}'_{\alpha,\beta} = \{E \in \mathrm{Coh}^\beta(X) \mid \text{any subobject } 0 \neq F \hookrightarrow E \text{ satisfies } \nu_{\alpha,\beta}(F) \leq 0\},$$

and set $\mathcal{A}^{\alpha,\beta}(X) = \langle \mathcal{F}'_{\alpha,\beta}[1], \mathcal{T}'_{\alpha,\beta} \rangle$. For any $s > 0$ we define

$$\lambda_{\alpha,\beta,s} = \frac{\mathrm{ch}_3^\beta - s\alpha^2 H^2 \cdot \mathrm{ch}_1^\beta}{H \cdot \mathrm{ch}_2^\beta - \frac{\alpha^2}{2} H^3 \cdot \mathrm{ch}_0^\beta}.$$

An object $E \in \mathcal{A}^{\alpha,\beta}(X)$ is called $\lambda_{\alpha,\beta,s}$-(semi)stable if for any nontrivial subobject $F \hookrightarrow E$ we have $\lambda_{\alpha,\beta,s}(F) < (\leq) \lambda_{\alpha,\beta,s}(E)$.

Note that $\mathrm{D}^b(X_2)$ has a full strong exceptional collection $(\mathcal{O}_{X_2}(-1), \mathcal{S}(-1), \mathcal{O}_{X_2}, \mathcal{O}_{X_2}(1))$, where $\mathcal{S}$ is a spinor bundle on X_2. The following results of Schmidt can be used for description of sheaves on X_2 with a given Chern character.

Proposition 6 ([30],[32, Thm. 6.1(2)]). *(i) Let $\alpha < \frac{1}{3}, \beta \in [-\frac{1}{2}, 0], s = \frac{1}{6}$. For any $\gamma \in \mathbb{R}$ we define a torsion pair*

$$\mathcal{T}''_\gamma = \{E \in \mathcal{A}^{\alpha,\beta}(X_2) \mid \textit{any quotient } E \twoheadrightarrow G \textit{ satisfies } \lambda_{\alpha,\beta,s}(G) > \gamma\},$$
$$\mathcal{F}''_\gamma = \{E \in \mathcal{A}^{\alpha,\beta}(X_2) \mid \textit{any subobject } 0 \neq F \hookrightarrow E \textit{ satisfies } \lambda_{\alpha,\beta,s}(F) \leq \gamma\}.$$

There is a $\gamma \in \mathbb{R}$ such that

$$\langle \mathcal{T}''_\gamma, \mathcal{F}''_\gamma[1] \rangle = \mathfrak{C} := \langle \mathcal{O}_{X_2}(-1)[3], \mathcal{S}(-1)[2], \mathcal{O}_{X_2}[1], \mathcal{O}_{X_2}(1) \rangle.$$

(ii) Let v be the Chern character of an object from $\mathrm{D}^b(X)$, and $\alpha_0 > 0, \beta_0 \in \mathbb{R}$, and $s > 0$ such that $\nu_{\alpha_0,\beta_0}(v) = 0$, $H^2 \cdot v_1^{\beta_0} > 0$, and $\Delta(v) \geq 0$. Let us assume that all ν_{α_0,β_0}-semistable objects of class v are ν_{α_0,β_0}-stable. Then there is a neighborhood U of the point (α_0, β_0) such that for all $(\alpha, \beta) \in U$ with $\nu_{\alpha,\beta}(v) > 0$, an object $E \in \mathrm{Coh}^\beta(X)$ with $\mathrm{ch}(E) = v$ is $\nu_{\alpha,\beta}$-semistable if and only if it is $\lambda_{\alpha,\beta,s}$-semistable.

6 Moduli of rank 2 sheaves on Fano threefolds

The first direction of research in our paper [40] concerns the question of the boundedness of the third Chern class c_3 of semistable rank 2 sheaves on X (as in the previous section, X is a rational Fano threefold of the main series) with fixed $c_1 \in \{-1, 0\}$ and $c_2 \geq 0$ and getting estimates for the third Chern class c_3. Using the tilt stability technique in the derived category $\mathrm{D}^b(X)$, we gave an almost complete answer to this question for the three-dimensional quadric X_2 in the following theorem (see paragraphs (3.1)-(4.2) in [40, Theorem 3.1]).

Theorem 4. *(i) Let E be a semistable sheaf on the quadric X_2 of rank 2 with $c_1 = -1$. Then $c_2 \geq 0$ and $c_3 \leq \frac{1}{2}c_2^2$ if c_2 is even, and, respectively, $c_3 \leq \frac{1}{2}(c_2^2 - 1)$ if c_2 is odd.*
(ii) Let E be a semistable sheaf of rank 2 on X_2 with $c_1(E) = 0$. Then $c_2 \geq 0$ and $c_3 \leq \frac{1}{2}c_2^2$, if c_2 is even, and, respectively, $c_3 \leq \frac{1}{2}(c_2^2 + 1)$, if c_2 is odd.
These estimates are exact for all $c_3 \geq 0$.

The proof of this theorem is based on the study of the relationship between tilt-semistability and Bridgland semistability in $\mathrm{D}^b(X_2)$. The key here is Schmidt's important technical result (2014) on the description of a subcategory in $\mathrm{D}^b(X_2)$ generated by a torsion pair, which we recalled in Proposition 6 (i).

Unfortunately, no analogues of this result are known to date for varieties X_4 and X_5. Therefore, for these varieties it is not possible to use the same method to obtain exact upper bounds for the class c_3 for all semistable sheaves of rank 2 on X_4 and X_5. However, using more traditional technique considering the behavior of stable sheaves at standard birational transformations $X_4 \dashrightarrow X_1$

and $X_5 \dashrightarrow X_2$, we give a partial answer to the question about boundedness of c_3 for a sufficiently wide class of sheaves on X_4 and X_5.

For $X = X_4$ or X_5 we denote by $B(X)$ the base of the family of lines on X. As is known, $B(X_4)$ is a smooth Abelian surface, and $B(X_5) \simeq \mathbb{P}^2$. Let us give the following definition.

Definition 11. *Reflexive sheaf E of rank 2 with first Chern class $c_1(E) = 0$ on $X = X_4$ or $X = X_5$ is called a sheaf of main type if for any line $l \in B(X)$ not passing through points from* Sing E *we have either $E|_l \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$, and such lines constitute a dense open set in $B(X)$, or $E|_l \cong \mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}(-m)$, where $m > 0$, and the set $B_2(X) := \{l \in B(X) \mid E|_l \cong \mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}(-m),\ m \geq 2\}$ has dimension ≤ 0.*

In [40, Theorem 4.4] we give examples of infinite series of components of moduli spaces of semistable sheaves in which the general sheaf is a reflexive sheaf of main type. (Presumably the property of being a sheaf of main type is true for all stable reflexive sheaves of rank 2 with $c_1 = 0$, that is, perhaps an analogue of the Grauert-Mülich theorem holds for them, which is known to be valid for stable reflexive sheaves of rank 2 on X_1.) For sheaves of main type we prove the following theorem (see [40, Theorem 6.4, Theorem 6.1]).

Theorem 5. *Let E be a stable reflexive sheaf of rank 2 of main type with Chern classes $c_1 = 0$, $c_2 > 0$, c_3 on the variety X_4 or X_5. Then the following inequalities are true for the class c_3 of the sheaf E.*
(i) On X_4: $c_3 \leq c_2^2 - c_2 + 2$.
(ii) On X_5: $c_3 \leq \frac{2}{9}c_2^2$ if c_2 is even, and, respectively, $c_3 \leq \frac{2}{9}c_2^2 + \frac{1}{2}$, if c_2 is odd. Whether these estimates are sharp is an open question.

The second direction of research in the article [40] is the construction of new infinite series (with growing class c_2) of moduli components of semistable sheaves of rank two on the varieties X_1, X_2, X_4 and X_5, including an explicit description of the general sheaves in these components. For $X = X_1$ several known series of moduli components were discussed before in this thesis. As for the varieties X_2, X_4 and X_5, before our work on each of them only one infinite series of moduli components of semistable sheaves of rank 2 was found. These are the series of components containing as open sets the families of instanton bundles. Instantonic bundles on X_2 were defined by L. Costa and R. M. Miro-Roig in [13] in 2009, and on X_4 and X_5 and other Fano varieties by A. Kuznetsov [24] in 2012 and D. Faenzi [15] in 2013. In work [15] D. Faenzi proved that families of instanton bundles on X_2, X_4 and X_5 are indeed open subsets of irreducible components of moduli spaces, which are reduced at a general point and have the expected dimension. In recent years, an extensive number of works were devoted to the study of instanton series of bundles, a review of which can be found, for example, in [3] and [12].

In our article [40] we constructed several new infinite series of irreducible rational components of moduli spaces of rank 2 semistable sheaves on the varieties X_1, X_2, X_4 and X_5. We described general sheaves in these components and proved their reflexivity, and also found dimensions of the constructed components. These results were proven in [40, Theorem 4.1, Theorem 4.1S, Theorem 4.2, Theorem 4.2S, Theorem 4.3]. They are collected in Theorem 6 given below.

Let us recall here that general sheaves in these components are described as extensions, in which the left term is either a twisted trivial rank two bundle, or a twisted spinor bundle on X_2, or a a twisted rank two sheaf F, which we describe below.

(I) In the case $X = X_1$ the sheaf F is a reflexive sheaf determined from the exact triple

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3}^{\oplus 3} \to F \to 0. \tag{9}$$

(II) In the case when $X = X_4$ is a complete intersection of a general pencil of hyperquadrics in $\mathbb{P}^5$, let $\mathbb{P}^1 \subset |\mathcal{O}_{\mathbb{P}^5}(2)|$ be the base of this pencil of quadrics, and let Γ be a hyperelliptic curve of genus 2, defined as the double covering $\rho: \ \Gamma \to \mathbb{P}^1$, branched at points corresponding to degenerate quadrics of the pencil. Let $\Gamma^* = \rho^{-1}(\mathbb{P}^{1*})$, where $\mathbb{P}^{1*} \subset \mathbb{P}^1$ is an open subset of nondegenerate quadrics of the pencil, and let $\Delta = \rho^{-1}(\mathbb{P}^1 \smallsetminus \mathbb{P}^{1*})$. Any point $y \in \Gamma^*$ corresponds to one of two series of generating planes on a non-degenerate 4-dimensional quadric $Q(y) := \rho(y)$, and this series corresponds to a spinor bundle $\mathcal{S}(y)$ of rank 2 on $Q(y)$ with $\det \mathcal{S}(y) = \mathcal{O}_{Q(y)}(1)$. In this case we set $F_y = \mathcal{S}(y)|_X$. Let now $y \in \Delta$, that is, the degenerate quadric $Q(y)$ is a cone with its vertex at the point say $z(y)$, so the projection $\mu : Q(y) \smallsetminus \{z(y)\} \to Q_y$ is defined, where Q_y is a smooth three-dimensional quadric. On Q_y the spinor bundle $\mathcal{S}_{Q_y}$ with $\det \mathcal{S}_{Q_y} = \mathcal{O}_{Q_y}(1)$ is defined, and we set $F_y = \mu^* \mathcal{S}_{Q_y}|_X$. The sheaf F in this case can be any one from the sheaves F_y for $y \in \Gamma$.

(III) In the case $X = X_5$, the sheaf F is defined as the restriction to X of the tautological bundle on the Grassmannian $\mathrm{Gr}(2, 5)$, twisted by $\mathcal{O}_X(1)$.

Theorem 6. *Let X be one of the varieties X_1, X_2, X_4, X_5, and let $\mathcal{O}_X(1)$ be an ample sheaf on X such that $\mathrm{Pic}(X) = \mathbb{Z}[\mathcal{O}_X(1)]$. Consider a sheaf E of rank 2 on X defined by one of nontrivial extensions of the form*

$$0 \to F_i \to E \to G_j \to 0, \qquad 1 \leq i \leq 3, \quad 1 \leq j \leq 2, \tag{10}$$

where $F_1 = \mathcal{O}_X(-n)^{\oplus 2}$, $F_2 = F(-n)$, where F is a rank 2 sheaf of one of the types (I)-(III) described above, $G_1 = \mathcal{O}_S(m)$, where $S \in |\mathcal{O}_X(k)|$, and the sheaves F_3 and G_2 are defined in the case of the quadric $X = X_2$, namely, $F_3 = \mathcal{S}(-n)$, where $\mathcal{S}$ is a spinor bundle on X_2 with $\det \mathcal{S} = \mathcal{O}_X(1)$, and $G_2 = \mathcal{J}_{\mathbb{P}^1,S}(m)$, where $S \in |\mathcal{O}_X(1)|$, $\mathbb{P}^1$ is a line on the surface S. Let $M_X(v)$ be the Gieseker-Maruyama moduli scheme of semistable sheaves on X with Chern character $v = \mathrm{ch}(E)$, determined from the triple (10), and let

$$M := \{[E] \in M_X(v) \mid E \text{ is a Gieseker stable extension } (10)\}. \tag{11}$$

Then the following statements are true.
(1) For $X_1,\ X_2,\ X_4,\ X_5$ in the case of $i = j = 1$, $k \geq 1$, $n = \lceil \frac{k}{2} \rceil$, $m < -n$,
(2) for $X_1,\ X_4,\ X_5$ in the case of $i = 2$, $j = 1$, $k \geq 1$, $n = \lfloor \frac{k}{2} \rfloor$, $m < -n$,
(3) for X_2 in each case
(3.1) $i = 1$, $j = 2$, $n = 1$, $m \leq -1$,
(3.2) $i = 3$, $j = 1$, $k \geq 1$, $n = \lfloor \frac{k}{2} \rfloor + 1$, $m \leq -n$,
(3.3) $i = 3$, $j = 2$, $n = 1$, $m \leq -1$,
the set M is a smooth dense open subset of an irreducible component $\overline{M}$ of the moduli scheme $M_X(v)$. Moreover, M is a fine moduli space, and reflexive sheaves form a dense open set in M. Moreover, all components $\overline{M}$ of infinite series (1), (2) and (3.1)-(3.3) are rational varieties for each of the varieties X_l, $l = 1, 2, 4, 5$, except for the series (2) for $X = X_4$, in which each component is irrational. Moreover, in all cases the dimensions of the components $\overline{M}$ are found as polynomials from $\mathbb{Q}[k, m, n]$ or $\mathbb{Q}[m]$ respectively.

A significant part of our article [40] is devoted to the research on semistable sheaves of rank 2 with maximal class c_3 on the quadric X_2. We show that for $c_1 \in \{-1, 0\}$ and all values of the class c_2, except for a few small values, every such sheaf is given by an extension of the form (10), that is, in the notation (11) we have equality $M = \overline{M}$. In this case the construction from the proof of Theorem 6 allows for a significant refinement, giving complete description of all moduli spaces of semistable sheaves with maximal class c_3 on X_2. In the remaining cases of small values of c_2 and maximal $c_3 \geq 0$ it is also possible to obtain an explicit description of moduli spaces, except for two cases, in which we only proved that these spaces are not smooth. In the case $(c_1, c_2) = (0, 1)$ we proved that the maximal value of c_3 of a semistable sheaf should be negative, but could not determine it precisely. These results, proven in [40, Theorems 5.1 — 5.4] are collected in the following two theorems.

Theorem 7. *Let $X = X_2$ be a quadric, and $M_X(v)$ be the moduli scheme of Gieseker-Maruyama semistable sheaves E of rank 2 on X with Chern classes (c_1, c_2, c_3), where $c_1 \in \{-1, 0\}$, $c_2 \geq 0$, $c_3 = c_{3\max} \geq 0$ is maximal for each c_2, and*

$$v = \mathrm{ch}(E) = (2, c_1 H, \frac{1}{2}(c_1^2 - c_2)H^2, \frac{1}{2}(c_{3\max} + \frac{2}{3}c_1^3 - c_1 c_2)[\mathrm{pt}]),$$

where $H = c_1(\mathcal{O}_X(1))$. Then the following statements hold.
(1.i) For $c_1 = -1$, even $c_2 = 2p$, $p \geq 2$, and $c_{3\max} = \frac{1}{2}c_2^2$ the variety $M_X(v)$ is a Grassmannization of 2-dimensional quotient spaces of the vector bundle of rank $\frac{1}{4}(c_2 + 2)^2$ on the space $\mathbb{P}^4$ defined by the first formula (38) in [40] for $n = 1$ and $m = -p$. In this case $\dim M_X(v) = \frac{1}{2}(c_2 + 2)^2$.
(1.ii) For $c_1 = -1$, odd $c_2 = 2p + 1$, $p \geq 1$, and $c_{3\max} = \frac{1}{2}(c_2^2 - 1)$ the variety $M_X(v)$ is the Grassmannization of 2-dimensional quotient spaces of the vector bundle of rank $\frac{1}{4}(c_2 + 1)(c_2 + 3)$ on the Grassmannian $\mathbb{G} = \mathrm{Gr}(2, 4)$ defined by the second formula (38) in [40] for $m = -p$. In this case $\dim M_X(v) = \frac{1}{2}(c_2 + 1)(c_2 + 3)$.
(1.iii) For $c_1 = 0$, odd $c_2 = 2p + 1$, $p \geq 1$, and $c_{3\max} = \frac{1}{2}(c_2^2 + 1)$ the variety $M_X(v)$ is a projectivization of the vector bundle of rank $\frac{1}{2}(c_2 + 1)(c_2 + 3)$ on the space $\mathbb{P}^4$, defined by the formula (61) in [40] with $n = 1$ and $m = -p$. In this case $\dim M_X(v) = \frac{1}{2}c_2^2 + 2c_2 + \frac{9}{2}$.
(1.iv) For $c_1 = 0$, even $c_2 = 2p$, $p \geq 3$, and $c_{3\max} = \frac{1}{2}c_2^2$ the variety $M_X(v)$ is a projectivization of the vector bundle of rank $\frac{1}{2}c_2^2 + 2c_2 + 1$ on the Grassmannian $\mathbb{G}$, defined by the formula (77) in [40] for $m = 1 - p$. In this case $\dim M_X(v) = \frac{1}{2}c_2^2 + 2c_2 + 4$.
(2) In all the above cases, the scheme $M_X(v)$ is irreducible and is a smooth rational projective variety, all sheaves from $M_X(v)$ are stable, the general sheaf in $M_X(v)$ is reflexive, and $M_X(v)$ is a fine moduli space.

Theorem 8. *Under the conditions and notation of Theorem 7, the following statements are true:*

(1) For $c_1 = -1$*,* $c_2 = 1$ *and* $c_{3\max} = 0$*, the variety* $M_X(v)$ *is a point* $[\mathcal{S}(-1)]$*.*
(2) For $c_1 = c_2 = c_{3\max} = 0$*, the variety* $M_X(v)$ *is a point* $[\mathcal{O}_X^{\oplus 2}]$*.*
(3) For $c_1 = -1$*,* $c_2 = 2$ *and* $c_{3\max} = 2$ *we have* $M_X(v) \simeq \mathrm{Gr}(2, 5)$*.*
(4) For $c_1 = 0$*,* $c_2 = 2$ *and* $c_{3\max} = 2$ *the scheme* $M_X(v)$ *is irreducible, has dimension 9 and is not smooth.*
(5) For $c_1 = 0$*,* $c_2 = 4$ *and* $c_{3\max} = 8$ *the scheme* $M_X(v) = M_X(2; 0, 4, 8)$ *is the union of two irreducible components* M_1 *and* M_2*. These components are described as follows.*
(5.i) M_1 *is a smooth rational variety of dimension 20, which is the projectivization of a locally free sheaf of rank 17 on Grassmannian* $\mathbb{G}$*.* M_1 *is a fine moduli space and all sheaves in* $M_X(v)_1$ *are stable. Moreover, the scheme* $M_X(v)$ *is nonsingular along* M_1*.*
(5.ii) the scheme M_2 *is irreducible, has dimension 21, and polystable sheaves in* M_2 *form a closed subset of dimension 12 in* M_2*, in which the scheme* $M_X(v)$ *is not smooth.*

We highlight the last statement (iii) of [40, Theorem 5.4] as a separate theorem due to its importance.

Theorem 9. *For the quadric* $X = X_2$*, the scheme* $M_X(2; 0, 4, 8)$ *is disconnected:*

$$M_X(2; 0, 4, 8) = M_1 \sqcup M_2,$$

and its irreducible components M_1 *and* M_2 *are described above in the statements (5.i)-(5.ii) of Theorem 8.*

This result gives the first example of a disconnected moduli scheme of semistable sheaves of rank two on a smooth projective 3-dimensional variety. In all the few known so far cases where the issue of connectedness of the module scheme $M_X(2; c_1, c_2, c_3)$ with fixed c_1, c_2, c_3 was discussed, the union of all known components of the module scheme turned out to be connected. In particular, in the work [21, Thm. 25, Thm. 27] connectedness of the scheme $M_{\mathbb{P}^3}(2; 0, 2, 0)$ was proved, as well as connectedness of the union of seven known by 2017 irreducible components of the scheme $M_{\mathbb{P}^3}(2; 0, 3, 0)$, and in the same place [21, Prop. 24] for an arbitrary positive value n connectedness of the union of some growing with n number of known components of $M_{\mathbb{P}^3}(2; 0, n, 0)$ was proved. In the work [1, Main Thm. 3] connectedness of the scheme $M_{\mathbb{P}^3}(2; -1, 2, m)$ for all admissible positive values of m, namely, for $m = 0,\ 2,\ 4$, was proved. To our opinion, one of the possible reasons for the disconnectedness of the scheme $M_{X_2}(2,0,4,8)$ in Theorem 9 can be the fact that the quadric X_2, unlike $\mathbb{P}^3$, is not a toric variety.

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